3.4 The Solow Model: Population Growth and Technological Progress

GDP

\[ Y_t = F(K_t, A_t N_t) \]

Labor efficiency

\[ A_t \]

Saving

\[ s Y_t \]

Consumption

\[ C_t = (1 - s) Y_t \]

Depreciation

\[ \delta K_t \]

Change of capital stocks over time:

\[ K_{t+1} - K_t = s Y_t - \delta K_t \]

- Population growth
  \[ N_{t+1} = (1+n) N_t \]

- Population growth rate
  \[ n \]

- Technological progress
  \[ A_{t+1} = (1+g) A_t \]

Rate of technological progress \( g \)
Savings rate $s = \frac{\text{Gross Investment}}{\text{GDP}}$

Sparquote der USA

Quelle: Table 5.1, Saving and Investment, Bureau of Economic Analysis, http://bea.gov/bea/dn/nipaweb/
Aggregate Savings
Net Investment / NDP

Sparquote der USA

# Savings Rates of Selected Regions

<table>
<thead>
<tr>
<th></th>
<th>World</th>
<th>USA</th>
<th>Euro Zone</th>
<th>Africa</th>
<th>Asian Emerging Markets</th>
<th>Middle East</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average of 1993-2000</td>
<td>22.1%</td>
<td>16.8%</td>
<td>21.4%</td>
<td>17.5%</td>
<td>32.9%</td>
<td>24.2%</td>
</tr>
<tr>
<td>2006</td>
<td>22.8%</td>
<td>13.7%</td>
<td>21.3%</td>
<td>24.8%</td>
<td>42.2%</td>
<td>40.4%</td>
</tr>
</tbody>
</table>

Source: IMF, July 2007
The Solow Model:
Population Growth and Technological Progress

Constant returns to scale => GDP per effective worker

\[ y_t = \frac{Y_t}{(A_t N_t)} = F \left( \frac{K_t}{(A_t N_t)}, 1 \right) = f(k_t) \]

- capital intensity \[ k_t = \frac{K_t}{(A_t N_t)} \]
- gross investment = saving \[ s \ y_t \]
- consumption \[ \frac{C_t}{(A_t N_t)} = c_t = (1 - s) \ y_t \]
- depreciation \[ \delta k_t \]

Change of capital intensity over time:

\[ k_{t+1} - k_t \approx \frac{sy_t - (\delta + g + n)k_t}{(1 + g)(1 + n)} \]

**Steady state** \( k^* \):

\[ s \ f(k^*) = (\delta + n + g) \ k^* \]
The Solow Model: Population Growth and Technological Progress

**Per capita magnitudes in steady state with technological progress**

\[
\frac{K_t}{N_t} = A_t k^*
\]

Per capita magnitudes of capital stock, output and consumption grow with the rate of technological progress in the long term.

\[
\frac{Y_t}{N_t} = A_t f(k^*)
\]

\[
(1-s) \frac{Y_t}{N_t}
\]

\[
s \frac{Y_t}{N_t}
\]
The Solow Model: Population Growth and Technological Progress

The capital intensity in steady state depends on $s$, $n$ and $g$

$$k^*(s, n, g) : \quad sf(k^*) = (\delta + n + g)k^*$$

- Total differentiation gives

$$sf'(k^*)dk^* = (\delta + n + g)dk^* + k^*dn$$

$$\Rightarrow \quad \frac{\partial k^*}{\partial n} = \frac{k^*}{sf'(k^*) - \delta - n - g} < 0$$

- equivalently

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{\delta + n + g - sf'(k^*)} > 0$$
The Solow Model: Decline in Population Growth

With a decline in growth of population, less investment is necessary to maintain capital intensity. Therefore, a constant savings rate leads to a higher capital intensity. Is the higher capital intensity optimal?
The Solow Model: Population Growth and Technological Progress

Golden Rule \[ \max_{k^*} f(k^*) - (\delta + n + g) k^* \]

- Optimization condition

\[ k^{**}: \quad f'(k^{**}) = \delta + n + g \]

\[ f' = \text{marginal product of capital} \]

\[ k^{**} = (f')^{-1}(\delta + n + g) \]

With \( f'' < 0 \), the capital intensity decreases when \( \delta, n \) and \( g \) increases.
The Solow Model: Decline in Population Growth

In this example with a constant savings rate, the capital intensity increases more strongly than it should (Golden rule $k^{**} < k^*$). A constant savings rate leads to overinvestment.
The Solow Model: Decline in Population Growth

On the one hand, capital intensity should increase, when \( n \) decreases.

On the other hand, the decline in population growth automatically leads to an increase in capital intensity with a constant saving rate.

How does the savings rate react to a decline in population growth?
The Solow Model: Decline in Population Growth

Comparative statistics:

Total differential of the equation

\[ f'(k^*(s^*,n,g)) = \delta + n + g \]

- gives

\[ f''(\cdot) \left[ \frac{\partial k^*}{\partial s} ds^* + \frac{\partial k^*}{\partial n} dn \right] = dn \]

- Inserting the formula from **Slide 54** gives

\[ f''(\cdot) \frac{f(\cdot) ds - k dn}{\delta + n + g - sf'(\cdot)} = dn \]

- \[ f''(\cdot) f(\cdot) ds - f''(\cdot) k dn = (\delta + n + g - sf'(\cdot)) dn \]
The Solow Model: Decline in Population Growth

\[
\frac{ds}{dn} = \delta + n + g - sf'(\cdot) + f''(\cdot)k
\]

- The denominator is negative. The numerator can be either positive or negative!

A clear answer to the question of whether the savings rate should rise or fall with a decline in \( n \) can be only reached with more information concerning the production function.

- When the saving rate cannot adjust, can \( k \) increase beyond the Golden Rule?

  ► Overinvestment!  ► Japan?
The Solow Model: Decline in Population Growth

Consumption per unit of labor efficiency with a decline of $n$ and a constant saving rate.

At period $t_0$, the growth rate of working population falls from $n_0$ to $n_1$. 

$c_0^* = (1-s) f(k_0^*)$

$c_1^* = (1-s) f(k_1^*)$
The Solow Model:
Population Growth and Technological Progress

Consumption per capita with a decline of \( n \) and a constant savings rate

\[
\frac{C_1^*}{N} = (1-s) A_t f(k_1^*)
\]

\[
\frac{C_0^*}{N} = (1-s) A_t f(k_0^*)
\]

At period \( t_0 \), the growth rate of working population falls from \( n_0 \) to \( n_1 \).
The Solow Model: Example

Example: \( f(k) = k^\alpha \quad 0 < \alpha < 1 \)

Steady state: \( sf(k) = (\delta + n + g)k \)

\[
\iff \quad sk^\alpha = (\delta + n + g)k \quad \iff \quad k = \left( \frac{s}{n + \delta + g} \right)^{\frac{1}{1-\alpha}}
\]

Golden Rule: \( f'(k) = \delta + n + g \)

\[
\iff \quad \alpha k^{\alpha-1} = \delta + n + g \quad \iff \quad k = \left( \frac{\alpha}{\delta + n + g} \right)^{\frac{1}{1-\alpha}}
\]
The Solow Model: Example

In steady state, the Golden Rule requires:

\[ k = \left( \frac{s}{n + \delta + g} \right)^{\frac{1}{1-\alpha}} = \left( \frac{\alpha}{n + \delta + g} \right)^{\frac{1}{1-\alpha}} \]

Hence it follows: \( s = \alpha \)

The production function \( f(k) = k^\alpha \) describes the limiting case where the optimal savings rate is independent of \( n \).
The Solow Model

The Solow model describes the optimal saving in steady state.

The adjustment process takes time though. The Solow model does not describe the optimal adjustment track.

The ‘optimal saving rate’ maximizes the per capita consumption in steady state. The steady state will never be completely reached.

Time preference: future consumption should be discounted. Consumption during the adjustment phase must be considered.

These critiques are considered by Ramsey model.

Recession studies: business cycles, growth and employment
Technological Progress

3.5. The role of technological progress in the process of growth

3.6. Determinants of the technological progress
3.6.a) Optimal patent protection

3.7. Distribution effects of technological progress

Literature:

- Blanchard, Chapter 12-13.
- Abel & Bernanke, Macroeconomics, 5th ed., Chapter 6
Technological Progress

Dimensions of technological progress:

- Higher productivity of the factors capital and labor
- Better products
- New products
- A larger variety of products
3.5 Growth and Technological Progress

Production function \( Y = F(K,AN) = K^a (AN)^{1-a} \)

- GDP growth rate \( \frac{dY_t}{Y_t} \)
- Rate of technological progress \( g = \frac{dA_t}{A_t} \)
- Growth rate of working population \( n = \frac{dN_t}{N_t} \)
- Per effective worker \( y = f(k) = k^a \)

Steady State (Solow): \( sf(k^*) = \delta + n + g \)

\( k \) converges to \( k^* \) and \( y \) to \( y^* \).
Growth and Technological Progress

In steady state, output per effective worker \( y = \frac{Y}{(AN)} \) is constant.

- GDP \( Y \) grows with rate \( n+g \).
- GDP per capita \( \frac{Y}{N} \) grows with rate \( g \).
- In the long run, the rate of technological progress alone determines the growth of material wealth.
## Growth and Technological Progress

<table>
<thead>
<tr>
<th></th>
<th>Growth of Output per Capita</th>
<th>Rate of Technological Progress</th>
</tr>
</thead>
<tbody>
<tr>
<td>France</td>
<td>4.0</td>
<td>1.8</td>
</tr>
<tr>
<td>Germany</td>
<td>4.9</td>
<td>2.1</td>
</tr>
<tr>
<td>Japan</td>
<td>8.0</td>
<td>3.1</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>2.5</td>
<td>1.8</td>
</tr>
<tr>
<td>United States</td>
<td>2.2</td>
<td>1.6</td>
</tr>
<tr>
<td>Average</td>
<td>4.3</td>
<td>2.1</td>
</tr>
</tbody>
</table>
# Growth and Technological Progress

## Spending on R&D as a Percentage of GDP

<table>
<thead>
<tr>
<th></th>
<th>1963</th>
<th>1975</th>
<th>1989</th>
</tr>
</thead>
<tbody>
<tr>
<td>France</td>
<td>1.6</td>
<td>1.8</td>
<td>2.3</td>
</tr>
<tr>
<td>Germany</td>
<td>1.4</td>
<td>2.2</td>
<td>2.9</td>
</tr>
<tr>
<td>Japan</td>
<td>1.5</td>
<td>2.0</td>
<td>3.0</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>2.3</td>
<td>2.0</td>
<td>2.3</td>
</tr>
<tr>
<td>United States</td>
<td>2.7</td>
<td>2.3</td>
<td>2.8</td>
</tr>
</tbody>
</table>

Source: Kumiharu Shigehara, “Causes of Declining Growth in Industrialized Countries.”
Growth and Technological Progress

Evidence:
Rate of technological progress has declined.
However, the fraction of expenditure on research and development has not declined.
Has the research process become inefficient?
Growth and Technological Progress

Errors in measurement:
Product quality and variety need findings from research and development that do not necessarily increase GDP with competitive prices.

Example: electronic equipments
Improvement in capacity with constant price.
Capacity characteristics do not agree with price comparison.
Consequences: Overestimation of inflation, underestimation of technological progress.
Growth and Technological Progress

Production function $Y = F(K, AN) = K^a (AN)^{1-a}$

- Total differential:
  
  $$dY = (AN)^{1-a} a K^{a-1} dK + (1-a) K^a (AN)^{-a} (A \, dN + N \, dA)$$

- $dY/Y = a \, dK/K + (1-a) (dN/N + dA/A)$

- $dY/Y = a \, dK/K + (1-a) \, n + (1-a) \, g$

- Growth rate of GDP is composed of proportions, that is, based on the growth of three factors: capital, labor and knowledge.

- What fraction does technological progress have?
**Growth and Technological Progress**

**Sources of Growth in USA (% per year)**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Labor growth</td>
<td>1.34</td>
<td>1.45</td>
</tr>
<tr>
<td>Capital growth</td>
<td>0.56</td>
<td>1.18</td>
</tr>
<tr>
<td>Productivity growth</td>
<td>1.02</td>
<td>0.97</td>
</tr>
<tr>
<td>Total output growth</td>
<td>2.92</td>
<td>3.60</td>
</tr>
</tbody>
</table>


Growth and Technological Progress

How do we measure technological progress?

1. \( \frac{dY}{Y} = a \frac{dK}{K} + (1-a) \ n + (1-a) \ g \)
2. \( \Rightarrow \ (1-a) \ g = \frac{dY}{Y} - a \frac{dK}{K} - (1-a) \ n \)
3. Solow residual growth in real terms
4. in nominal terms:
   \( \Rightarrow \ (1-a) \ g = \frac{dY^{nom}}{Y^{nom}} - \pi - a ( \frac{dK^{nom}}{K^{nom}} - \pi ) - (1-a) \ n \)
   \( = \frac{dY^{nom}}{Y^{nom}} - a \frac{dK^{nom}}{K^{nom}} - (1 - a) (n + \pi) \)
5. Overestimation of inflation = Underestimation of technological progress
3.6 Determinants of Technological Progress

Technological progress is not exogenous.

Where does technological progress arise from?

How is the cost-benefit analysis of the decision maker?

Is the level of research and development efficient?
Determinants of Technological Progress

Knowledge as a public good

- Non-rivalrous
- Non-excludable

- In the economic process, research and development (R&D) provide competition.

Patents establish property rights on processes and product innovations.

Assumption: profit motive leads to microeconomic optimal expenditure on R & D.
Determinants of Technological Progress

Knowledge as a public good

R&D has positive external effects on macro economy. Research results of one institute help research of another institute.

=> positive external effects

- Private supply of public goods leads to undersupply, because individual decision maker does not internalize the external effects.
Determinants of Technological Progress

**Microeconomic solution:** provide subsidies to private suppliers or provide as public goods through the government.

- **Advantages of private supply:**
  Competitive and efficient input of resources.
  Efficient in terms of firms’ profit.

- **Disadvantages:**
  Focusing on firms’ or macroeconomic profit does not always result in the same research direction.
  Privatization of results hinder spillovers. Subsidies become entitlements.
Determinants of Technological Progress

**Advantages of public supply:** Adjustment of research to maximize social welfare. Strong spillovers (e.g. between research and education).

**Disadvantages:** little efficiency control.

**Conclusion:** coexistence of both systems and utilization of their respective advantages through task sharing.
3.6.a) Optimal Patent Protection

Product innovations are directly evaluated on the market. Clear measure of value.

- Patents hinder product competition and lead to higher price, lower consumer surplus, and monopolistic profit of firms. => inefficiency

At the same time, monopolistic profit stimulates firms to invest in R&D.

- Optimal patent protection must balance the trade-off between positive effects on stimulation of R&D, and welfare gains from spillover effects.
Optimal Patent Protection

Optimal patents: A simple partial equilibrium model

A firm decides how much to spend on R&D today to bring a new product to the market. The probability for successful development is $\pi(R)$, where $R$ denotes the current expenditure on R&D.
Optimal Patent Protection

The demand for the new product is: \( x = A - b \ p \)

The marginal cost is constant = c.

\[ \text{Market situation during patent protection} \]

\( \text{CRM} = \text{consumer surplus in monopoly} \)

\( \text{CR}_M = \text{consumer surplus in monopoly} \)

\( p_M = \text{Monopolistic price} > c \)

\[ \text{Profit} \]

\[ \text{Welfare loss} \]

\[ \text{p} \]

\[ \text{p}_M \]

\[ \text{c} \]

\[ \text{X}_M \]

\[ \text{X} \]
Optimal Patent Protection

After expiration of the patent (after T periods), competition arises: the price sinks to marginal cost

\[ p = c \]

Market situation after expiration of patent protection
Optimal Patent Protection

Firms decide on research expenditure $R$ and maximize their present value of future profits:

The longer the patent protection lasts, the longer the profit accumulates, the higher is the corresponding expected earnings from research.

Optimal research expenditure depends positively on the duration of patent protection: $R(T), \quad R' > 0$

But: longer patent protection also results in welfare losses accruing for a longer period.

- The government decides on patent right $T$ and maximizes the macroeconomic benefits.

Secondary condition: $R = R(T) \quad \text{Firms’ decision!}$
Optimal Patent Protection

**Notice:** The optimal patent protection balances the trade-off between:

- Welfare loss that is caused by the monopolistic market: The longer the patent protection, the higher the loss.

- Welfare gain that arises as expected monopolistic profits stimulate innovations. Under short-term patent protection private R&D are not attractive. Innovations don’t occur!
3.7 Technological Progress and Income Distribution

Two perceptions:

Technological progress enhances output and thereby salaries.

Process innovations set work force free and therefore degrade the wage rate in market equilibrium.

Different types of technological progress
Technological Progress and Income Distribution

Production function \( Y = F(K,AN) = K^a \ (AN)^{1-a} \)

- Wage according to marginal productivity
- Wage \( w = \text{marginal product of labor} = \frac{dF}{dN} \)
  \[ w = (1-a) \ K^a \ A^{1-a} \ N^{-a} \]
- Interest \( r = \text{marginal product of capital} = \frac{dF}{dK} \)
  \[ r = a \ K^{a-1} \ (AN)^{1-a} \]

- Income distribution by Cobb Douglas production function:
  \[ w \ N = (1-a) \ Y, \quad r \ K = a \ Y \]

Factor incomes increase with GDP
Technological Progress and Income Distribution

Functional income distribution

With Cobb Douglas production function the wage rate is:

\[ w \frac{N}{Y} = (1 - a) \text{ constant} \]

- Define „Hicks neutral technological progress“:
  With constant capital intensity the wage rate stays constant.

- \( \Rightarrow \) Technological progress has a proportional effect on both factors.

In steady state the wage rate stays constant.
Technological Progress and Income Distribution

Labor-saving technological progress:
○ Marginal product of capital increases more than marginal product of labor.

Wage rate decreases.

○ Capital-saving technological progress:

Marginal product of capital declines more than marginal product of labor.

Wage rate increases.
Technological Progress and Income Distribution

Labor saving technological progress:
Marginal product of capital increases more than marginal product of labor.

Wage rate decreases.
It allows wages to increase when the marginal product of labor increases less than the marginal product of capital.

However, there can also be a decrease in the marginal productivity of the factor labor.
Then not only does the wage rate declines, but the real wage w as well.
Technological Progress and Income Distribution

**Notice:** Technological progress can have distributional effects when the marginal productivity of different factors increases in different sizes.

Rationalized investment works without input of low-skilled labor and causes decline in low wage.

**Empirical:** increasing wage inequality
## Wage Inequality

### Real Wage Changes for Full-Time Workers 1963 - 1995 (%)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>All Workers</strong></td>
<td>17.7</td>
<td>-11.2</td>
</tr>
<tr>
<td>By education (years of schooling)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0-11 (less than high school)</td>
<td>17.2</td>
<td>-20.2</td>
</tr>
<tr>
<td>12 (high school)</td>
<td>18.8</td>
<td>-13.4</td>
</tr>
<tr>
<td>13-15 (less than 4 years of college)</td>
<td>17.7</td>
<td>-12.4</td>
</tr>
<tr>
<td>16+ (4 years of college or more)</td>
<td>18.9</td>
<td>3.5</td>
</tr>
<tr>
<td>18+ (graduate school)</td>
<td>25.8</td>
<td>14.0</td>
</tr>
<tr>
<td>By sex</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Men</td>
<td>18.3</td>
<td>-17.4</td>
</tr>
<tr>
<td>Women</td>
<td>16.8</td>
<td>-1.5</td>
</tr>
</tbody>
</table>

Source: Lawrence Katz and David Autor, “Changes in the Wages Structure and Earnings Inequality”
Technological Progress and Income Distribution

Reasons for increasing wage inequality:

1. Globalization (Heckscher Ohlin Samuelson theorem):
   The wages of qualified labor become equal with free capital flow.

In emerging markets, the fraction of low-skilled labor is larger than in developed countries.

International competition squeezes the wages, particularly those of low-skilled workers.

Lecture in international economics, recession studies
Technological Progress and Income Distribution

Reasons for increasing wage inequality:

2. Skill-biased technological progress

New production technologies require a high fraction of skilled labor.

When demand for skilled labor increases, demand for low-skilled labor declines.

When the education system can not make the fraction of skilled labor increase to meet demand, there is a relative shortage skilled labor.
Technological Progress

Summary:

In the long run, growth is determined solely by the rate of technological progress.

Measurement of the rate of technological progress does not calculate productivity improvement correctly and hence underestimate the rate.

Technological progress presumes research and development.

R & D is a public good. In market equilibrium R & D is too low.
Technological Progress

Summary (2):

Patent rights stimulate R&D, however, they hinder the efficient utilization of research results.

Technological progress in general leads to increases in factor incomes.

Globalization and knowledge-based technological progress enhance wage inequality.